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R-NILPOTENCY IN HOMOTOPY EQUIVALENCES†

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ABSTRACT

We study the monoid of self homotopy equivalences of an R-nilpotent space, with the goal of understanding the actions of a cyclic group of order p on a simply-connected homologically finite space with uniquely p-divisible homotopy groups.

§1. Introduction

Let p be a prime and let R be the ring $\mathbf{Z}[1/p]$. The aim of this note is to use the ideas of Dror and Zabrodsky [DZ] to study the topological monoid $\mathrm{Aut}^h X$ of homotopy self-equivalences of a connected R-nilpotent space X [BK: III §5] under the assumption that X is homologically finite-dimensional. The main technical result (§2) states that the classifying spaces of certain natural submonoids of $\mathrm{Aut}^h X$ are themselves R-nilpotent.

There are two direct applications. Let σ be a cyclic group of order p with chosen generator s. Any basepoint-preserving map $f\colon B\sigma \to B\,\mathrm{Aut}^h X$ gives rise to a self-equivalence $f(s)$ of X as well as to an induced automorphism $f(s)_*$ of the integral homology H_*X.

1.1. Proposition. *Assume that X is a connected R-nilpotent CW-complex such that H_iX vanishes for sufficiently large i. Let $f, g\colon B\sigma \to B\,\mathrm{Aut}^h X$ be two basepoint-preserving maps such that $f(s)_* = g(s)_*$. Then f and g are freely homotopic, i.e., homotopic through a homotopy that does not necessarily preserve basepoints.*

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1.2. REMARK. The homotopy of 1.1 can be chosen so that it carries the basepoint of $B\sigma$ to a loop in $B\,\mathrm{Aut}^h X$ which represents a self-equivalence of X inducing the identity map on homology (§3). It seems difficult in general to do better than this.

1.3. PROPOSITION. *Let X be as in* 1.1 *and let $t: X \to X$ be a homotopy equivalence with the property that $(t^p)_* - 1$ is a nilpotent endomorphism of H_*X. Then there exists a basepoint-preserving map $f: B\sigma \to B\,\mathrm{Aut}^h X$ such that*

(i) *$f(s): X \to X$ commutes up to homotopy with t, and*

(ii) *$f(s)_* - t_*$ is a nilpotent endomorphism of H_*X.*

1.4. REMARKS. In 1.3, t^p stands for the p-fold composite of t with itself. Conditions (i) and (ii) actually determine f up to free homotopy (§3). Note (3.1) that if $(t^p)_*$ is the identity map of H_*X then $f(s)_*$ equals t_*.

1.5. REMARKS. Propositions 1.3 and 1.1 can be interpreted, respectively, as existence and uniqueness results for actions of σ on spaces homotopy equivalent to X [DDK].

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§2. The main result

Let X be a connected R-nilpotent CW-complex such that H_iX vanishes for sufficiently large i. An *R-flag* Φ for H_*X is a collection of finite R-submodule filtrations

$$0 \subseteq \Phi_{i,n_i} \subseteq \cdots \subseteq \Phi_{i,1} \subseteq \Phi_{i,0} = H_iX$$

of the integral homology groups of X (recall that H_iX is isomorphic to $H_i(X, R)$ [BK: V, §3]). Any self-equivalence $t: X \to X$ induces maps $t_*: H_iX \to H_iX$; we will say that t is *upper triangular* with respect to Φ if $t_*(\Phi_{i,j}) = \Phi_{i,j}$ $(\forall\, i, j)$ and that t is *strictly upper triangular* with respect to Φ if in addition the induced automorphisms of the quotients $\Phi_{i,j}/\Phi_{i,j+1}$ are identity maps. Let $\mathrm{Aut}^h(X, \Phi)$ denote the submonoid of $\mathrm{Aut}^h X$ consisting of all self-equivalences which are strictly upper triangular with respect to Φ.

The goal of this section is to prove the following proposition.

2.1. PROPOSITION. *Let X be a connected R-nilpotent CW-complex such that H_iX vanishes for sufficiently large i, and let Φ be an R-flag for H_*X. Then the classifying space $B\,\mathrm{Aut}^h(X, \Phi)$ is R-nilpotent.*

2.2. Lemma. *Let G be a nilpotent group and M an R-module upon which G acts nilpotently. Then there is a unique nilpotent action of $G \otimes R$ on M extending the given action of G. Moreover, any map $f: M \to M'$ of R-modules with nilpotent G-action respects the extended nilpotent action of $G \otimes R$.*

Remarks. Here $G \otimes R$ denotes the R-localization of the nilpotent group G [BK: V, §2].

Proof of 2.2. Let S denote the semi-direct product of G with M. It follows from naturality and the fiber lemma [BK: II, 4.8] that the unique nilpotent action of $G \otimes R$ on M extending the given action of G is the one provided by the action of $G \otimes R$ on the first homology group of the fiber in the localized fibration sequence

$$K(M, 1) \to R_\infty K(S, 1) \to R_\infty K(G, 1) \approx K(G \otimes R, 1).$$

The functoriality property of the extended action is immediate.

2.3. Remark. The uniqueness and functorality provisions of 2.2 imply that if G respects a given filtration of M and acts trivially on the associated graded, then the same is true of $G \otimes R$.

Proof of 2.1. Let

$$X \longrightarrow U \xrightarrow{q} B\,\mathrm{Aut}^h(X, \Phi)$$

be the fibration over $B\,\mathrm{Aut}^h(X, \Phi)$ associated to the action of $\mathrm{Aut}^h(X, \Phi)$ on X. The fibration q is universal for fibrations $E \to B$ with fiber X such that the monodromy action of $\pi_1 B$ on $H_* X$ is strictly upper triangular with respect to Φ [DZ, 4.2]. The space $B\,\mathrm{Aut}^h(X, \Phi)$ is nilpotent [DZ, 3.5]. The fibration q is nilpotent ([BK: II, §4], [DZ, §3]) and so by [BK: II, 4.2] there is a localized fibration sequence

$$X \longrightarrow R_\infty U \xrightarrow{R_\infty q} R_\infty B\,\mathrm{Aut}^h(X, \Phi)$$

which is also nilpotent; in particular, in the fibration $R_\infty q$ the fundamental group of the base acts nilpotently on the homology of the fiber. By 2.3, then, this monodromy action of $\pi_1 R_\infty B\,\mathrm{Aut}^h(X, \Phi) = (\pi_1 B\,\mathrm{Aut}^h(X, \Phi)) \otimes R$ on $H_* X$ is strictly upper triangular with respect to Φ (i.e. the action must be the canonical extension (2.2) of the nilpotent action of $\pi_1 B\,\mathrm{Aut}^h(X, \Phi)$ on $H_* X$). It follows that $R_\infty q$ is classified by a map $R_\infty B\,\mathrm{Aut}^h(X, \Phi) \to B\,\mathrm{Aut}^h(X, \Phi)$. The composite of this classifying map with the localization map $B\,\mathrm{Aut}^h(X, \Phi) \to$

$R_\infty B\,\mathrm{Aut}^{\mathrm{h}}(X, \Phi)$ is clearly homotopic to the identity. Thus $B\,\mathrm{Aut}^{\mathrm{h}}(X, \Phi)$, as a homotopy retract of the R-nilpotent space $R_\infty B\,\mathrm{Aut}^{\mathrm{h}}(X, \Phi)$, is itself R-nilpotent.

§3. Two applications

In this section σ will denote a cyclic group of order p with chosen generator s. The following lemma can be proved either by a direct obstruction-theory argument or by an application of the homotopy inverse limit spectral sequence of [BK: XI, §7].

3.1. Lemma [DwZ, 2.3]. *Let*

$$F \longrightarrow E \xrightarrow{q} B\sigma$$

be a fibration sequence in which the fiber F is connected and R-nilpotent. Then the space of sections of q is connected and R-nilpotent.

Proof of 1.1. Let $\mathrm{Aut}^{\mathrm{h}}(X, H_*X)$ denote the submonoid of $\mathrm{Aut}^{\mathrm{h}}\,X$ consisting of self-equivalences which act as the identity on H_*X, and let G denote the discrete quotient group $\pi_0\,\mathrm{Aut}^{\mathrm{h}}(X)/\pi_0\,\mathrm{Aut}^{\mathrm{h}}(X, H_*X)$. There is a fibration sequence

$$B\,\mathrm{Aut}^{\mathrm{h}}(X, H_*X) \longrightarrow B\,\mathrm{Aut}^{\mathrm{h}}\,X \xrightarrow{q} BG$$

in which the fiber is connected and R-nilpotent (2.1). It follows from 1.1 that if $f, g: B\sigma \to B\,\mathrm{Aut}^{\mathrm{h}}\,X$ are basepoint-preserving maps such that $q \cdot f$ is homotopic as a pointed map to $q \cdot g$, then f and g are themselves homotopic by a homotopy which leads the basepoint of $B\sigma$ through a loop in the fiber $B\,\mathrm{Aut}^{\mathrm{h}}(X, H_*X)$. This completes the proof.

3.2. Lemma. *Let M be an R-module and $t: M \to M$ an automorphism with the property that $t^p - 1: M \to M$ in a nilpotent endomorphism. Then there exists a unique automorphism s of M such that*

(i) *s commutes with t,*

(ii) *$s^p = 1$, and*

(iii) *$s - t: M \to M$ is a nilpotent endomorphism.*

Proof. The action of t on M gives a map $R[T, T^{-1}] \to \mathrm{End}_R(M)$ sending T to the automorphism t; since $t^p - 1$ is nilpotent, this factors through a homomorphism $h: R[T, T^{-1}]/(T^p - 1)^k \to \mathrm{End}_R(M)$ for some $k > 0$. By Hensel's lemma there is a unique element S of $R[T, T^{-1}]/(T^p - 1)^k$ such that

$S^p = 1$ and S is congruent to T modulo $(T^p - 1)$; let $s = h(S)$. It is clear that s satisfies (i)–(iii) and has the additional advantage of being a polynomial in t and t^{-1}. Let s' be any other automorphism of M satisfying (i)–(iii). Since s' commutes with s and t, the difference element $s' - s = (s' - t) + (t - s)$ is a nilpotent endomorphism of M. Write

$$(s')^p = (s + (s' - s))^p = s^p + ps^{p-1}(s' - s) + \cdots$$

so that

$$0 = (s' - s)\left(ps^{p-1} + \binom{p}{2} s^{p-2}(s' - s) + \cdots\right).$$

The second factor on the right-hand side of the last equation is an invertible endomorphism of M, since ps^{p-1} is invertible and the rest of the sum consists of a nilpotent endomorphism that commutes with s. It follows that $s' - s$ is zero.

Proof of 1.3. Let Φ be the R-flag for H_*X given by the images of powers of $t^p - 1$, so that t is upper triangular with respect to Φ and t^p is strictly upper triangular. Let $\mathrm{Aut}^h(X : \Phi)$ denote the monoid of self-equivalences of X which are upper triangular with respect to Φ and let G be the discrete quotient group $\pi_0 \mathrm{Aut}^h(X : \Phi)/\pi_0 \mathrm{Aut}^h(X, \Phi)$. There is a fibration sequence

$$B\,\mathrm{Aut}^h(X, \Phi) \longrightarrow B\,\mathrm{Aut}^h(X : \Phi) \xrightarrow{\;q\;} BG$$

in which the fiber is R-nilpotent (2.1). The hypotheses of the proposition give a commutative diagram

$$\begin{array}{ccc} B\mathbf{Z} & \xrightarrow{\;\tau\;} & B\,\mathrm{Aut}^h(X : \Phi) \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ B\sigma & \longrightarrow & BG \end{array}$$

in which q' carries the generator x of $\mathbf{Z}$ to s and $\tau_\#$ carries x to t. Now perform fiberwise localization [BK: I, §8] on q and on q'; this has no effect on q because the fiber of q is R-nilpotent. What results is a diagram

$$\begin{array}{ccc} \dot{R}_\infty(B\mathbf{Z}) & \longrightarrow & B\,\mathrm{Aut}^h(X : \Phi) \\ \downarrow & & \downarrow \\ B\sigma & \longrightarrow & B\sigma \end{array}$$

in which the left-hand vertical arrow is a principal fibration with fiber $R_\infty B\mathbf{Z} \approx B\mathbf{Z}[1/p]$. Since $H^2(B\sigma, \mathbf{Z}[1/p])$ vanishes, it is clear that the space $\dot{R}_\infty(B\mathbf{Z})$ is actually a product $B\sigma \times B\mathbf{Z}[1/p]$. Let $f: B\sigma \to B\,\mathrm{Aut}^{\mathrm{h}} X$ be the composite

$$B\sigma \to \dot{R}_\infty(B\mathbf{Z}) \to B\,\mathrm{Aut}^{\mathrm{h}}(X : \Phi) \to B\,\mathrm{Aut}^{\mathrm{h}} X.$$

Conditions (i) and (ii) follow immediately. The uniqueness property 1.4 of f follows from 3.2 and 1.1.

References

[BK] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. **304**, Springer-Verlag, Berlin, 1972.

[DDK] E. Dror, W. Dwyer and D. M. Kan, *Equivariant maps which are self homotopy equivalences*, Proc. Am. Math. Soc. **4** (1980), 670–672.

[DwZ] W. G. Dwyer and A. Zabrodsky, *Maps between classifying spaces*, in *Algebraic Topology, Barcelona 1986*, Lecture Notes in Math. **1298**, Springer-Verlag, Berlin, 1987, pp. 106–119.

[DZ] E. Dror and A. Zabrodsky, *Unipotency and nilpotency in homotopy equivalences*, Topology **18** (1979), 187–1976.